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CRACK GENERATION AND PROPAGATION MECHANISM

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UDC 539.3

As the load increases, a solid body goes over from one stable equilibrium position to another. The dynamic opening of cracks can be treated as buckling with a subsequent jump-like passage to a new stable position. Consequently, the mathematical methods of studying the equilibrium, stability, and buckling of mechanical systems, in particular the apparatus of catastrophe theory, can be used to analyze the carrying capacity, crack formation and propagation, and rupture. As has been noted in [1], the descending branches of the laws of system element interaction are here of substantial value since they permit the determination of all the possible equilibrium positions.

In this paper, an approach based on studying equilibrium positions by catastrophe theory methods is used to analyze the behavior of some of the simplest discrete models of a solid in the form of atomic lattices subjected to load. The process of crack formation and opening is represented graphically up to rupture.

1. Let us first examine a model representing several parallel series of atoms (Fig. 1a). The interaction force between the first two series, referred to the unit of length of these series is given by the function [1]

$$\Phi = Ex \exp(-x/x_f),$$

where x is the magnitude of the change in the distance between series, x_f is the value of x corresponding to the maximal force, and E is Young's modulus. In contrast to [1], we assume that in removing the load the interaction force changes according to the linear law $E(x - x_e)$ for fixed x_e. Here x_H is the inelastic component of the displacement x, $x_H = x[1 - \exp \times (-x/x_f)]$. The interaction force between the series b and d referred to the length unit is determined by the expression $\Phi' = Ec^{-1}y$, where y is the magnitude of the change in spacing, and c is a numerical parameter taking account of the pliability of the system of atomic series between b and d. The higher such series, the higher the value of c.

Keeping the series a fixed, we stretch the system quasistatically by giving a displacement u to the series d (stiff loading). The strain potential energy here is

$$\Pi = \int_0^x \Phi dx + (u - x)^2 E/2c.$$

It is natural to call the variable x the state parameter and u, c the control parameters. Then the function Π can be considered as a two-parameter family of functions $\Pi: S \times C$, where $S = R$ is the space of states $(x \in S)$, $C = R \times R = R^2$ is the control space $((u, c) \in C)$, and R is the set of real numbers.

The critical points of the function Π are determined by the equation

$$\partial\Pi/\partial x = Ex \exp(-x/x_f) - (u - x)E/c = 0. \quad (1.1)$$

For these values of (u, c) all the equilibrium positions of the system evidently are obtained by solving (1.1). The set of these solutions, the points (x, u, c) form a manifold of the catastrophe M in three-dimensional space [2]. It has the form of a surface with a build-up and is shown in Fig. 1b. Here and henceforth $x_f = 0.4$.

The doubly and triply degenerate critical points are determined form the joint solution of (1.1) and [3]

$$\partial^2\Pi/\partial x^2 = E(1 - x/x_f) \exp(-x/x_f) + E/c = 0 \quad (1.2)$$

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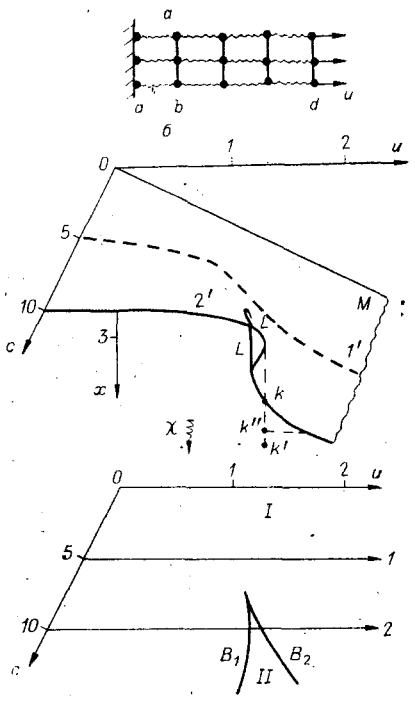

Fig. 1

(the Hessian of the function Π equals zero)

$$\partial^3\Pi/\partial x^3 = (E/x_f)(x/x_f - 2)\exp(-x/x_f) = 0. \quad (1.3)$$

Solving (1.1)-(1.3) we find the triply degenerate critical point ($x = 2x_f$, $u = 4x_f$, $c = c^2$) the beginning of the build-up, and from (1.1) and (1.2) a singular set of the manifold M consisting of doubly degenerate points [x, $u = x^2(x - x_f)^{-1}$, $c = x_f(x - x_f)^{-1}\exp(-x/x_f)$] that form the curve of the fold L (Fig. 1b).

The transform of the singular set to map the catastrophe M in the control space $\chi: M \to C$, projecting the points of M on the uc plane according to the rule $(x, u, c) \to (u, c)$, is called a bifurcation set B and is the place where the number and nature of the critical points change [2]. In this case it consists of the curves B_1 and B_2 (Fig. 1b).

Let us set $c = 5$. Then the loading path at C is entirely in the domain I exterior with respect to the set B (Fig. 1b, line 1). Just one sheet of the surface M, that the nondegenerate critical points corresponding to the stable equilibrium positions of the system form, will lie above the points of the domain I. Consequently, during loading the system goes smoothly over from one stable position to another (curve 1') until rupture, separation of the atomic series a and b to such a distance that the interaction force between them is negligibly small.

Now let $c = 10$. In this case the path in C (the line 2 in Fig. 1b) starts to traverse the domain I, and the path traversed by the equilibrium position in M and lying above it in C (curve 2' in Fig. 1b) is on the upper sheet of the surface M. We then drop into the domain II above which there are three sheets and three equilibrium positions, respectively: the unstable ones correspond to points of the surface M on the middle sheet within the curve of the fold, and the stable ones to points outside the curve of the fold. However, in conformity with the delay principle [2], the path in M is on the upper sheet until it no longer passes through the fold, i.e., does not drop in at the degenerate critical point and the sheet on which it is found does not "vanish." This happens for $u_* = 1.94$. If u_* is later disturbed by adding the term γu_*, then for arbitrarily small $\gamma > 0$ a nondegenerate critical point appears that already lies on the lower sheet of the surface M. Then the path in M jumps from one sheet of the surface to another. Such jumps are called catastrophic [2]. They occur when the path in C leaves the domain II (intersects the curve B_2) and smooth changes in control cause discontinuous changes of state.

The energy in the system drops during the jump. Consequently, in inertia it skips the equilibrium position lying on M (the point k with the coordinates $x = 1.696$, $u = 1.94$, Fig. 1b), and remains there where the energy level is restored (the point k' with the coordinates

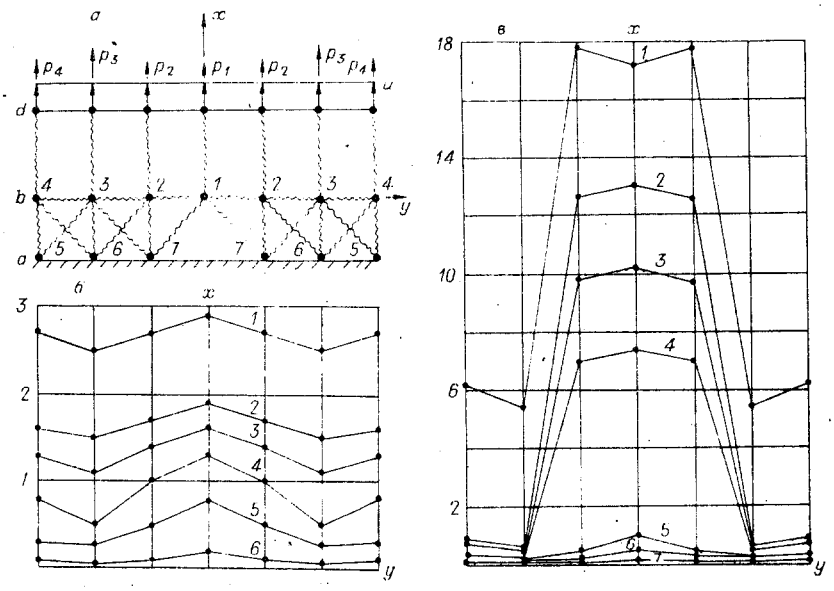

Fig. 2

x = 2.244, u = 1.94). Hence, only the series b moves. Then the system again tends to an equilibrium position. However, the interaction force between the atoms a and b now changes according to a linear unloading law. Then (1.1) takes the form

$$E(x - x_н) - E/c(u - x) = 0,\ x_н = 2.236.$$

Hence, the equilibrium position that the system tends to occupy, is determined by the point k" (x = 2.209, u = 1.94) that does not lie on the manifold M. Near this position, oscillations occur that damp out because of the internal friction in the system. Now computing the forces we find that the tensile forces are replaced by a compression force 0.0268E because the jump.

Since the series a and b interact according to the law of the descending branch of the curve Φ after the jump, it can then be considered that a sudden disturbance of the prolateness occurred [1] a crack was formed between whose edges a weak interaction is however conserved.

If the loading is continued, then k between u_* and $u = x_н$ the system is unloaded. Then a loading occurs during which the interaction forces are linear, and for u = 2.326 we again emerge on the manifold M. Later the deformation proceeds smoothly to final rupture.

Remark 1. The crack is propagated spontaneously if the quantity of elastic energy being liberated exceeds the energy being consumed in its growth, and stops when the magnitudes of these energies are equal (the reading is made from the start time). For instance, the elastic energy at the point k diminished by $0.0088E\ell^3$ (ℓ is the unit of length), and the expenditures in crack formation are $0.0785E\ell^3$, i.e., the crack continues to grow. The stop occurs at the point k' where the mentioned quantities are commensurate.

Remark 2. An analogous investigation is possible even for soft loading. In this case the system behavior is described by the total potential energy function, where the control parameters c and P are equivalent uniformly distributed forces applied to the atoms of the series d while the state parameters are u and x. Rupture always occurs for $P > \max \Phi$, where the parameter u certain reaches that magnitude for which an abrupt drop in the strain potential energy occurs in the system for $c > e^2$ during rupture. Hence, the total, and therefore the kinetic energy grow explosively, rupture is accelerated sharply.

2. As the second model we consider a simplified atomic lattice in which there is a defect, there is no central atom in the series a (Fig. 2a). There are n identical atomic series between the series b and d. We assume that the force needed to separate the atoms in the series b and d are opposite each other at a distance v (measuring from the initial state), and referred to the length unit is

$$\Phi_1 = 0{,}72(n+1)^{-1}E(v - v_н),\ v_н = v[1 - \exp(-v/(n+1))].$$

For the force of the interaction between atoms 1–2, 2–3, 3–4, 4–5, 3–6, 2–7 let us suggest the expression

$$\Phi_2 = E(w - w_н),\ w_н = w[1 - \exp(-2{,}5w)].$$

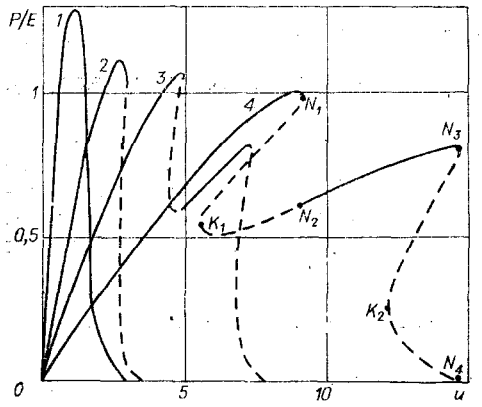

Fig. 3

The interaction between the atoms 1–7, 2–6, 3–7, 3–5, 4–6 will be weaker since the spacing between them in the initial equilibrium position is greater than between the atoms 1–2, 2–7, etc. Let us determine its function

$$\Phi_3 = E(z - z_H),\ z_H = z[1 - \exp(-5z)].$$

Here w, z are the elongations of the respective interatomic spacings, and v_H, w_H, z_H are the inelastic components of the elongation.

Keeping the series a fixed, we stretch the lattice by giving identical displacements u to the atoms in the series d. To simplify the calculations without distorting the qualitative pattern of the system behavior, we assume that the atoms can be displaced only in the vertical direction. Then the expression for the strain potential energy has the form

$$\Pi_1 = 2\left(\int_0^{m_1} + \int_0^{m_2} + 2\int_0^{m_3} + \int_0^{m_4}\right)\Phi_3 dz + \left(\int_0^{u-x_1} + 2\int_0^{u-x_2} + 2\int_0^{u-x_3} + 2\int_0^{u-x_4}\right)\Phi_1 dv + 2\left(\int_0^{x_2} + \int_0^{x_3} + \int_0^{x_4} + \int_0^{m_5} + \int_0^{m_6} + \int_0^{m_7}\right)\Phi_2 dw,$$

where x_i are the displacements of the appropriate atoms

$$m_i = \sqrt{(1 + x_i)^2 + 1} - \sqrt{2}\ (i = 1, 2, 3, 4);$$
$$m_j = \sqrt{(x_{j-4} - x_{j-3})^2 + 1} - 1\ (j = 5, 6, 7).$$

Here the control parameters are u and n and the state parameters are x_i.

The manifold of the catastrophe M_1 is determined by a system of four nonlinear algebraic equations obtained by equating the partial derivatives of the function Π_1 with respect to x_i to zero. The solution for each n fixed preliminarily and changing value of u is sought by successive approximations. Grouping all the nonlinear terms in the right sides and denoting them by f_i, we find the solution x_i^0 for $f_i = 0$. We then evaluate the quantity $f_i = f_i^0(x_i^0)$ and solve the linear system when $f_i = f_i^0$. We again calculate the quantity $f_i^0(x_i^0)$ for values obtained for x_i^i, etc. The sequence x_i^k (k = 0, 1, 2,...) converges to the desired solution. In conclusion we determine the elongations of all the interatomic spacings.

An analogous procedure is realized for all values of u. The elongations of the interatomic spacings are compared each time with the corresponding quantities obtained from the preceding solution. If it turns out that the spacing diminishes between some atoms, then the interaction force between them is later characterized by the unloading line. Consequently, it is necessary to find the specific value of the inelastic component corresponding to elongation of the spacing between these atoms, to substitute it in the expression for f_i and afterwards to set about solving the system for the next value of u.

As a result of the calculations, equilibrium positions are found by means of whose change the crack opening process can be traced. It is shown in Fig. 2b and c, respectively, for n = 1, u = 3, 2, 1.75, 1.5, 1.17, 0.5 (lines 1-6), and for n = 31, n = 14.8, 14.78, 12, 9.26, 9.24, 8, 4 (lines 1-7).

The manifold M_1, constructed in the three-dimensional space nuP, is a surface with two build-ups. Here P is equivalent to the tensile forces p_i applied to atoms of the series d displaced a distance u. Curves of the system equilibrium are displayed in Fig. 3 for n = 1, 7, 15, 31 (lines 1-4) obtained by a section through the manifold M_1 by the appropriate planes.

3. Starting from the results obtained, it can be concluded that a crack is generated at the site of a defect. When there is no internal defect, it occurs on the surface since the surface layer is weakened by its nature [4] (there is no series of bonds). The nature of the crack opening depends on the system pliability going over to the load in the zone of the defect. If it is slight (e.g., for n = 1), then the crack opens gradually (Fig. 2b), if it is large (n = 31), then the equilibrium opening is replaced by a jump (Fig. 2c). This deduction is in agreement with the result of experimental investigations on the influence of loading system pliability (the elastic energy reserve, respectively) on the kinetics of the strain and rupture process [5, 6].

Let us note that the loading path in M_1 does not intersect the fold in the first case (n = 1) but intersects in the second (n = 31), and the jump occurs from the upper to the lower sheet of the surface (from the point N_1 to the point N_2, Fig. 3). At this time the crack opens suddenly and certain quantity of energy ($0.91E\ell^3$) is liberated. The so-called snapping occurs [7]. Later, the equilibrium nature of opening is again replaced by a jump when the loading path intersects the second fold (a jump from the points N_3 and N_4). At this time the whole system ruptures.

Starting from other considerations, an analogous nature of crack behavior is noted in [8].

Remark 3. The parts of the curves that are constructed approximately are shown dashed in Fig. 3. The equality of the magnitude of the energy drop in the system during the jump and the area bounded, respectively, by the curve $N_1K_1N_2$ and the line N_1N_2, the curve $N_3K_2N_4$ and the line N_3N_4 (Fig. 3, n = 31) was used. A construction was also performed for other values of the parameter n.

During the jump the system skips the equilibrium position at M_1 and stops, when the energy level is restored. In other words, the crack is propagated spontaneously until the quantity of elastic energy being liberated is cancelled by the energy consumed in its growth (Griffith criterion). The magnitudes of the coordinates here reach the values $x_1 = 12.56$, $x_2 = 12.2$ (n = 31), the bonds 1–7, 2–6 are ruptured while 2–3, 2–7 are attenuated substantially.

After the jump the system evidently tends to return to the equilibrium position (the crack starts to close). Consequently, the bonds 3–7, 3–6, 3–5, 2–7, 2–3 go over into the compressed state. A process therefore occurs that is similar to the origination of residual compressible stresses in the neighborhood of the crack apex in an overload during cyclic loading, and they tend to close it [7]. Further loading does not result in opening of the crack until the appropriate compression forces are again replaced by tensile forces.

Let us also note that a the crack opens, the maximum force goes over into the undamaged zone. After a jump opening, compressive stresses occur at the crack edges since the bond 1–2 turns out to be in the compressed state (after the second jump so does bond 2–3 also). And, finally, computations show the drop in system carrying capacity as its size and pliability in increase (see Fig. 3).

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DETERMINATION OF STRESS INTENSITY FACTORS FOR CRACKS OF COMPLEX SHAPE IN ANISOTROPIC PLATES

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The application of analytical methods to the problem of fatigue crack propagation and branching is complicated by the shortage of information on the stress distribution near the tip of cracks of complex configuration. A discussion of this problem and a survey of the studies in this area can be found in [1], for example. Below we develop a method of solving a problem concerning a system of cracks of complex form in an anisotropic half-plane. An efficient algorithm for numerical solution of the problem is proposed. A study is made of the effect of anisotropy of the material, the free edge of the plate, and the curvature of the crack on the stress intensity factors at the tips of the cracks.

1. We will examine an elastic plate made of a homogeneous anisotropic material occupying the region $D = \{x > 0\}$. The plate is weakened by smooth, curved, non-intersecting internal notches L_j $(j = 1, \ldots, k)$, and is subjected to a system of external forces. We will assume that the edges of the notches are free of loads and are not in contact with one another.

The stresses in the plate are expressed through two analytic functions [2]:

$$(\sigma_x, \tau_{xy}, \sigma_y) = 2\,\mathrm{Re}\left\{\sum_{\nu=1}^{2}(\mu_\nu^2, -\mu_\nu, 1)\Phi_\nu(z_\nu)\right\},\quad z_\nu = x + \mu_\nu y \quad (\nu = 1, 2), \tag{1.1}$$

where μ_ν are the roots of the characteristic equation.

We seek the unknown functions $\Phi_\nu(z_\nu)$ in the form

$$\Phi_\nu(z_\nu) = \Phi_\nu^0(z_\nu) + \Phi_\nu^1(z_\nu),$$
$$\Phi_\nu^1(z_\nu) = \frac{1}{2\pi i}\int_L\left\{\frac{\omega_\nu(\tau)\,d\tau_\nu}{\tau_\nu - z_\nu} - \frac{l_\nu s_\nu\overline{\omega_1(\tau)}\,d\overline{\tau}_1}{\overline{\tau}_1 - s_\nu z_\nu} - \frac{n_\nu m_\nu\overline{\omega_2(\tau)}\,d\overline{\tau}_2}{\overline{\tau}_2 - m_\nu z_\nu}\right\}. \tag{1.2}$$

Henceforth, we use the notation in [3]; $\Phi_\nu^0(z_\nu)$ is the solution for the half-plane without notches from the prescribed system of external forces. The values of $\Phi_\nu^0(z_\nu)$ will be assumed to be known. The functions $\Phi_\nu^1(z_\nu)$ were given in another form in [4].

The functions $\Phi_\nu(z_\nu)$ determined by Eqs. (1.2) satisfy the prescribed system of external forces, including the boundary conditions on the edge of the plate $x = 0$ and at infinity.

Inserting the limiting values of $\Phi_\nu(z_\nu)$ from (1.2) into the boundary conditions for L and parameterizing the contours $L_j = \{t = \tau^j(\xi);\ |\xi| < 1\}$, we obtain the following system of singular integral equations of the problem [3] to determine the unknown complex functions $\omega_\nu(t) = \{\omega_{\nu j}(t) | t \in L_j;\ j = 1, \ldots, k\}$:

$$\int_{-1}^{1}\frac{F_j(\xi, \eta)\,\chi_j(\eta)\,d\eta}{\eta - \xi} + \sum_{s=1}^{k}\int_{-1}^{1}\left[k_1^{js}(\xi, \eta)\,\chi_s(\eta) + k_2^{js}(\xi, \eta)\,\overline{\chi_s(\eta)}\right]d\eta = f_j(\xi),$$

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